Europ. J. Combinatorics (1980) **1**, 43–46

Some Applications of Ramsey's Theorem to Additive Number Theory

P. ERDÖS

About 50 years ago, Sidon called a sequence of integers $A=\{a_1<a_2<\cdots\}$ a $B_r^{(k)}$ sequence if the number of representations of n as the sum of r or fewer a's is at most k and for some n is exactly k. In particular he was interested in $B_2^{(1)}$, or, for short, B_2 sequences. For a B_2 sequence the sums a_i+a_j are all distinct. In 1933 Sidon asked me to find a B_2 sequence for which a_n increases as slowly as possible. I observed that the greedy algorithm immediately gives that there is a B_2 sequence for which

$$a_n<cn^3 \tag{1}$$

holds for every n. I also proved that for every B_2 sequence

$$\limsup_{n\to\infty} a_n/n^2=\infty. \tag{2}$$

Turán and I [3] showed that there is a B_2 sequence for which

$$\liminf_{n\to\infty} a_n/n^2<\infty. \tag{3}$$

There is a big gap between (1) and (2). It seemed likely that there is a B_2 sequence for which

$$a_n<n^{2+\varepsilon} \tag{4}$$

holds for every $n>n_0(\varepsilon)$, but the proof or disproof of (4) is nowhere in sight. Rényi and I proved by probabilistic methods that there is a $k=k(\varepsilon)$ for which there is a $B_2^{(k)}$ sequence satisfying (4).

First of all I wanted to show that there is a B_2 sequence for which $a_n=o(n^3)$. Very recently Ajtai, Komlós and Szemerédi by a deep and ingenious application of combinatorial analysis to number theory proved the existence of such a B_2 sequence. But their result falls far short of (4) and only gives

$$a_n<n^3/(\log n)^\alpha.$$

A few years ago Donald Newman and I (independently of each other) asked: Is there a $B_2^{(k)}$ sequence which is not the union of a finite number of B_2 sequences? We both expected that such a $B_2^{(k)}$ sequence will exist. I wanted to attack the problem by probabilistic methods. In our proof of (4) for $B_2^{(k)}$ sequences with Rényi we built our sequence by choosing n with probability $n^{-\frac{1}{2}-\delta}$ and then easily proved that for suitable δ almost all such sequences satisfy (4) and have property $B_2^{(k)}$. I wanted to show that almost all of these sequences are not the union of a finite number of B_2 sequences. This is almost certainly true and would be interesting for its own sake but I have not been able to prove it. Recently I observed that our conjecture with Newman follows easily from Ramsey's theorem. In fact I prove the following slightly stronger

THEOREM 1. *There is a $B_2^{(3)}$ sequence A so that if $A=\bigcup_{i=1}^T A_i$ is any decomposition of A as the union of a finite number of subsequences then at least one of the A_i is again a $B_2^{(3)}$ sequence.*

0195–6698/80/010043+04$01.00/0

Let $n_1 < n_2 < \cdots$ satisfy $n_{i+1}/n_i \geq 4$; in particular we can take $n_i = 4^i$. Our $B_2^{(3)}$ sequence A will be the integers of the form $n_i + n_j$, $i \neq j$. The inequality $n_{i+1}/n_i \geq 4$ implies that the integers of this form are all distinct and in fact every integer is the sum of distinct n's in at most one way. Denote by $f(m)$ the number of solutions of $m = a_i + a_j$. Observe that if m is the sum of four distinct n's $n_i + n_j + n_r + n_s$ then $f(m) = 3$, if $m = 2n_i + n_r + n_s$ or $2n_i + 2n_j$, then $f(m) = 1$ and for all other integers $f(m) = 0$. Thus our A has property $B_2^{(3)}$. Now if we decompose A into the union of finitely many sequences A_r, $r = 1, \ldots, T$, then this can be interpreted as the colouring of the edges of a complete graph of infinitely many vertices by T colours. (The vertices of our graph are the n_i, the edges the $n_i + n_j$, i.e., the elements of A, the edges of the rth colour are the numbers in A_r). Now by Ramsey's theorem there is a monochromatic complete graph, i.e. one of the A_r's contains all the numbers of the form $\{n_i + n_j\}$ for some infinite subsequence of the n's. In other words A_r has property $B_2^{(3)}$—as stated. Thus Theorem 1 is proved.

CONJECTURE. For every k there is a $B_2^{(k)}$ sequence A so that if $A = \bigcup_{r=1}^{T} A_r$ then at least one of the A_r's is a $B_2^{(k)}$ sequence.

THEOREM 1′. *Our conjecture holds for $k = 3$, all $k = 2^s$, and all $\frac{1}{2}\binom{2s}{s}$, $s = 1, 2, \ldots$.*

For $k = 3$ we already proved Theorem 1′. For $k = 2$ let A consist of the integers of the form $\{n_i + n_j\}$, $i \not\equiv j \pmod 2$. Clearly A is $B_2^{(2)}$. Theorem 1′ now follows from the well known result that if the edges of an infinite complete bipartite graph are coloured by a finite number of colours then there always is a monochromatic C_4.

If $k = 2^s$, $s > 1$, then A consists of the integers of the form $n_{i_1} + n_{i_2} + \cdots + n_{i_{s+1}}$ where the i_r, $r = 1, \ldots, s+1$, form a complete set of residues (mod $s+1$). If $k = \frac{1}{2}\binom{2s}{s}$ then A consists of all integers which are the sum of s distinct n's. Theorem 1′ then easily follows by Ramsey's theorem for s-tuples or for $k = 2^s$ by a result of mine [2].

These methods can no doubt be applied for other values of k too, but it is doubtful if it will work for every k. In particular I cannot at present prove my conjecture for $k = 5$.

More generally I conjecture that for every k and r there is a sequence A which has property $B_r^{(k)}$ and if we decompose A into the union of finitely many subsequences $\{A_s\}$, $1 \leq s \leq T$, then at least one of them again has property $B_r^{(k)}$. We can prove this by the simple methods used here for every r and infinitely many k.

Now we outline the proof of a set theoretic result: let $c > \aleph_1$. Then there is a set S of real numbers, $|S| = \aleph_2$, so that the number of solutions of (α is an arbitrary real number)

$$x + y = \alpha, \qquad x \in S, \qquad y \in S$$

is at most two and if we decompose S into the union of denumerably many subsets $S = \bigcup_{n=1}^{\infty} S_n$ then for at least one n there is an α_n for which the number of solutions of $\alpha_n = x + y$, $x, y \in S_n$ is two.

The proof follows almost immediately from a result of Hajnal and myself: let $|A| = \aleph_2$, $|B| = \aleph_1$, $A \cap B = \phi$, $A \cup B$ rationally independent. It is clear that if $c > \aleph_1$ such A and B exist. S now is the set of numbers $x + y$, $x \in A$, $y \in B$. If $\alpha = x_1 + x_2 + y_1 + y_2$, $x \in A$, $y \in B$ then the number of solutions of $\alpha = u + v$, $u, v \in S$ is two, by the rational independence of $A \cup B$ it can never be more than two. Now put $S = \bigcup_{n=1}^{\infty} S_n$. This induces a decomposition of the edges of the complete bipartite graph $K(A, B)$, $|A| = \aleph_2$, $|B| = \aleph_1$, into countably many classes. An old theorem of Hajnal and myself states that at least one of these classes, say S_n, contains a C_4 which shows that there is an α_n for which the number of solutions of $\alpha_n = u + v$, $u, v \in S_n$ is two—as stated.

Finally we state a few extremal problems. Let $1 \leqslant a_1 < \cdots < a_l \leqslant n$ be a finite B_2 sequence. Put $\max l = f(n)$. Turán and I proved

$$f(n) = (1 + o(1))n^{\frac{1}{2}}$$

and we conjecture that

$$f(n) = n^{\frac{1}{2}} + o(1). \tag{5}$$

(5) if true is probably very deep. I often offered \$500 for a proof or disproof.

Let $u_1 < \cdots < u_n$ be any set of n integers. Denote by H_n the largest r for which there always is a subsequence $u_{i_1} < \cdots < u_{i_r}$, $r = H_n$, for which the sums of any two are distinct. I conjectured that

$$H_n \geqslant (1 + o(1))n^{\frac{1}{2}}. \tag{6}$$

Komlós, Sulyok and Szemerédi [4] in a remarkable paper proved a general theorem which implies

$$H_n > cn^{\frac{1}{2}} \tag{7}$$

where c is an absolute constant independent of n and of the sequence U. Their method does not seem suitable to give (6).

Let $u_1 < \cdots < u_n$ be a sequence of integers with property $B_2^{(k)}$. $H_n^{(k)}$ is the largest integer for which one can always select a B_2 subsequence $u_{i_1} < \cdots < u_{i_l}$, $l = H_n(k)$. It seems likely that

$$\lim H_n^{(k)}/n^{\frac{1}{2}} = \infty. \tag{8}$$

I have not been able to prove (8), though it is not impossible that even $H_n^{(k)} > n^{\frac{1}{2}+c}$ holds for some $c > 0$. I can only give an upper bound for $H_n^{(k)}$.

THEOREM 2

$$H_n^{(2)} < cn^{\frac{3}{4}}, \qquad H_n^{(4)} < cn^{\frac{2}{3}}. \tag{9}$$

The proof uses the same method as Theorem 1 and 1′. Our sequence $u_1 < \cdots < u_n$, $n = m^2$ are the integers of the form

$$4^i + 4^j, \qquad 0 \leqslant i < 2m, \qquad 1 \leqslant j < 2m+1, \quad i \text{ even}, j \text{ odd}.$$

We observed in Theorem 1′ that our sequence satisfies $B_2^{(2)}$. Its terms can be represented by the edges of a complete bipartite graph of m white and m black vertices. The white vertices are the integers 4^{2i}, $i = 0, \ldots, m-1$ and the black vertices 4^{2j+1}, $j = 0, \ldots, m-1$. A well known theorem due to W. Brown, V. T. Sós, A. Rényi and myself [1] implies that every subgraph having $c_1 m^{\frac{3}{2}} = c_2 n^{\frac{3}{4}}$ edges contains a C_4, i.e. the corresponding subsequence cannot have property B_2 which proves the first inequality of (9).

To prove the second inequality of (9) let our sequence $u_1 < \cdots < u_n$, $n = m^3$ be the integers of the form

$$\{4^i + 4^j + 4^k\}, \quad i = 3t, \quad j = 3t+1, \quad k = 3t+2, \quad 0 \leqslant t < m. \tag{10}$$

These integers have property $B_2^{(4)}$. To complete our proof of (9) we show that any subsequence of Cm^2 terms cannot be a B_2 sequence.

To see this let $u_1, \ldots, u_t$, $t = Cm^2$ be a subsequence of the the integers (10). Denote by $\alpha(j, k)$ the number of indices i for which $4^i + 4^j + 4^k$ is one of our u's. Clearly

$$\sum_{1 \leqslant j,k \leqslant 3m} \alpha_{j,k} = t = Cm^2. \tag{11}$$

From (12) we obtain that there are two distinct pairs $\{j_1, k_1\}$, $\{j_2, k_2\}$ for which there are

$$\sum_{1 \leq j,k \leq n} \binom{\alpha_{j,k}}{2} > \binom{m}{2}. \tag{12}$$

From (12) we obtain that there are two distinct pairs $\{j_1, k_1\}$, (j_2, k_2) for which there are two i's i_1 and i_2 so that all the four numbers

$$4^{i_1}+4^{j_1}+4^{k_1}, \qquad 4^{i_1}+4^{j_2}+4^{k_2}, \qquad 4^{i_2}+4^{j_1}+4^{k_1}, \qquad 4^{i_2}+4^{j_2}+4^{k_2} \tag{13}$$

are u's. The sum of the first and fourth integer in (13) equals the sum of the second and third. Thus our subsequence is not a B_2 sequence, which completes the proof of Theorem 2. This proof could easily be reformulated in the language of hypergraphs.

Perhaps a further development of this method will show that for every $\varepsilon > 0$ there is a $k_0 = k_0(\varepsilon)$ such that

$$H_n^{(k)} < n^{\frac{1}{2}+\varepsilon}. \tag{14}$$

I could not decide (14)—in any case I feel fairly sure that (8) is true.

Note added in proof. Our conjecture has recently been proved for every k by J. Nešetřil and V. Rödl.

References

1. W. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9** (1966), 281–285; P. Erdös, A. Rényi and V. T. Sós, On a problem of graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 215–235.
2. P. Erdös, On extremal problems of graphs and generalized graphs, *Israel J. Math.* **2** (1964), 183–190.
3. All the references to B_2 sequences and the probabilistic method in number theory can be found in H. Halberstam and K. F. Roth, *Sequences*, Clarendon Press, Oxford (1966), Chapters 2 and 3; A. Stöhr, Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe I und II, *J. reine angew. Math.* **194** (1955), 40–65 and 111–140.
4. J. Komlós, M. Sulyok and E. Szemerédi, Linear problems in combinatorial number theory, *Acta Math. Hung. Acad. Sci.* **26** (1975), 113–121.

(Received 19 September 1979)

PAUL ERDÖS
Mathematical Institute, Hungarian Academy of Science,
Realtanoda utca 11–13, Budapest 5, Hungary